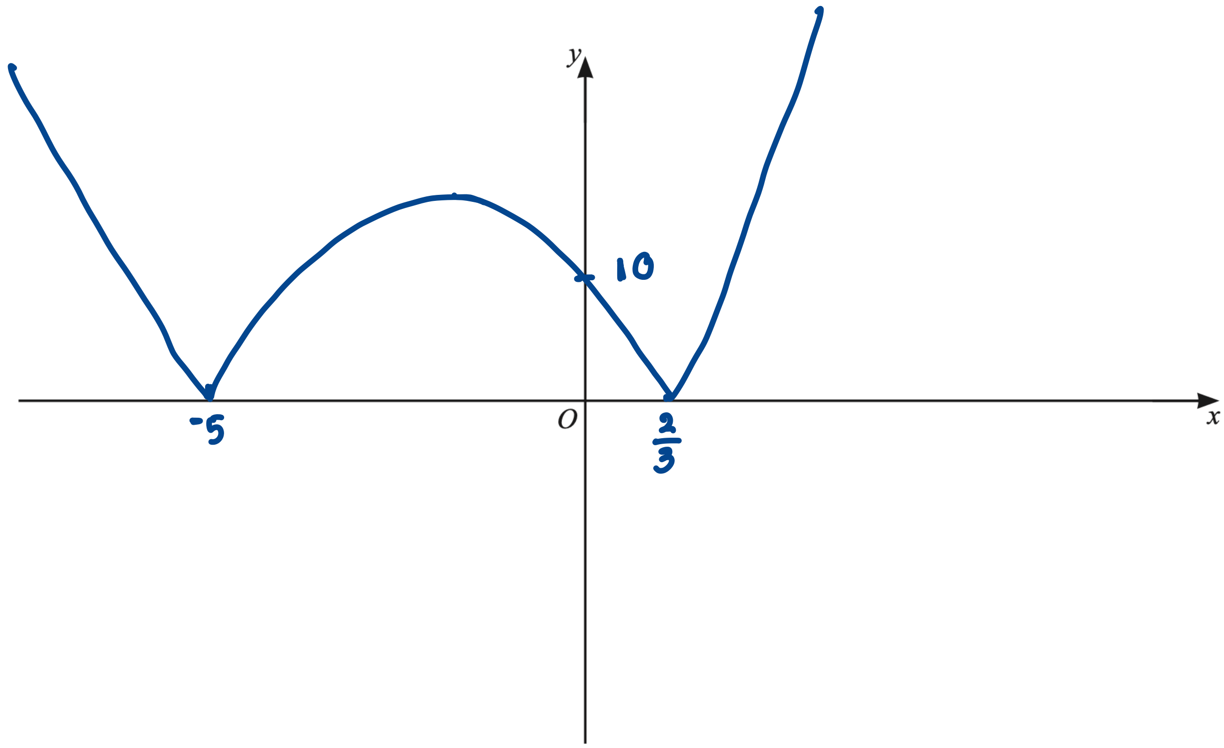
Chapter 1 to 9, 11a Test

/80 marks

1. (a) On the axes, draw the graph of $y = \left|3x^2 + 13x - 10\right|$, stating the coordinates of the points where the graph meets the axes.

[4]



(b) Find the set of values of the constant *k* such that the equation $kx^2 - 3(k + 1)x + 25 = 0$ has equal roots.

[4]

$a = k, \ b = -3k - 3, \ c = 25$

$b^2 - 4ac = 0$

$9k^2 + 18k + 9 - 100k = 0$

$9k^2 - 82k + 9 = 0$

$(9k - 1)(k - 9) = 0$

$k = 1/9 \quad \text{or} \quad k = 9$

2. (a) Solve the following simultaneous equations.

$$3y - 2x + 2 = 0$$
$$xy = \frac{1}{2}$$

[3]

$y = \frac{1}{2x}$

$\frac{3}{2x} - 2x + 2 = 0$

$3 - 4x^2 + 4x = 0$

$4x^2 - 4x - 3 = 0$

$(2x-3)(2x+1) = 0$

$x = \frac{3}{2}$ or $x = -\frac{1}{2}$

$y = \frac{1}{2 \times \frac{3}{2}}$ $\quad$ $y = \frac{1}{2 \times -\frac{1}{2}}$

$= \frac{1}{3}$ $\quad$ $= -1$

(b) Solve the equation $lg\,(2x - 1) + lg\,(x + 2) = 2 - lg\,4$.

[5]

$\lg (2x-1)(x+2) + \lg 4 = 2$

$\lg (2x^2 + 4x - x - 2) \times 4 = 2$

$\lg (2x^2 + 3x - 2) \times 4 = 2$

$8x^2 + 12x - 8 = 100$

$8x^2 + 12x - 108 = 0$

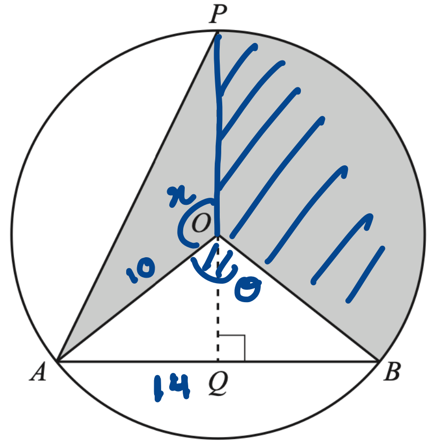
$\div 4$

$2x^2 + 3x - 27 = 0$

$x = 3$ or $x = -9/2$

(reject)

3.



The diagram shows a circle, centre *O*, radius 10 cm. The points *A*, *B* and *P* lie on the circumference of the circle. The chord *AB* is of length 14 cm. The point *Q* lies on *AB* and the line *POQ* is perpendicular to *AB*.

a. Show that angle *POA* is 2.366 radians, correct to 3 decimal places. [2]

$14^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \times \cos\theta$

$196 = 200 - 200\cos\theta$

$200\cos\theta = 200 - 196$

$\cos\theta = 1/50$

$\theta = 1.5508\text{rad}$

$\angle POA = \frac{2\pi - 1.5508}{2}$

$= 2.3661$

≈ 2.366 (3 d.p)

b. Find the area of the shaded region. [3]

$\text{sector} = \frac{1}{2}r^2\theta = \frac{1}{2} \times 100 \times \cancel{2.366}\ 1.183$

$= 118.3\text{ cm}^2$

$\Delta = \frac{1}{2}ab\sin c$

$= \frac{1}{2} \times 10 \times 10 \times \sin 2.366$

$= 35.007\text{ cm}^2$

total shaded $= 153\text{ cm}^2$

c. Find the perimeter of the shaded region. [5]

$AP^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \times \cos 2.366$

$AP^2 = 100 + 100 - 200 \cos 2.366$

$AP = 18.5$ cm

$BP = r\theta$

$= 10 \times 2.366$

$= 23.66$

shaded $P = 23.66 + 18.5 + 10 + 10$

$= 62.16$ cm

4. When e^{2y} is plotted against x^2, a straight line graph passing through the points (4, 7.96) and (2, 3.76) is obtained.

a. Find y in terms of x.

$$m = \frac{3.76 - 7.96}{2 - 4} = 2.1$$

[5]

$$e^{2y} = 2.1x^2 + c$$

$$3.76 = 4.2 + c$$

$$c = -0.44$$

$$e^{2y} = 2.1x^2 - 0.44$$

$$2y = \ln(2.1x^2 - 0.44)$$

$$y = \frac{1}{2}\ln(2.1x^2 - 0.44)$$

b. Find y when $x = 1$.

$$y = \frac{1}{2}\ln(2.1x^2 - 0.44)$$

[2]

$$= \frac{1}{2}\ln(2.1 - 0.44) = 0.253$$

c. Using your equation from **part (a)**, find the positive values of x for which the straight line exists.

$$y = \frac{1}{2}\ln(2.1x^2 - 0.44)$$

[3]

$$2.1x^2 - 0.44 > 0$$

$$2.1x^2 > 0.44$$

$$x^2 > 0.21$$

$$x > 0.458$$

5. The first four terms in ascending powers of *x* in the expansion $(3 + ax)^4$ can be written as $81 + bx + cx^2 + \frac{3}{2}x^3$. Find the values of the constants *a*, *b* and *c*.

[6]

$3^4 + {}^4C_1 (3)^3 (ax) + {}^4C_2 (3)^2 (ax)^2 + {}^4C_1 (3)^1 (ax)^3 + \dots$

$81 + 108ax + 54a^2x^2 + 12a^3x^3 + \dots$

$108a = b \qquad 54a^2 = c \qquad 12a^3 = \frac{3}{2}$

$b = 54 \qquad c = \frac{27}{2} \qquad a^3 = \frac{1}{8}$

$a = \frac{1}{2}$

6.The points *P* and *Q* have coordinates (5, -12) and (15, -6) respectively. The point *R* lies on the line *l*, the perpendicular bisector of the line *PQ*. The *x*-coordinate of *R* is 7.

a. Find the *y*-coordinate of *R*. [4]

$$m = \frac{-12+6}{5-15} = \frac{6}{10} = \frac{3}{5}$$

$$m_{\perp} = -\frac{5}{3}$$

$$\text{midpt} = (10, -9)$$

$$y = -\frac{5}{3}x + c$$

$$-9 = -\frac{50}{3} + c$$

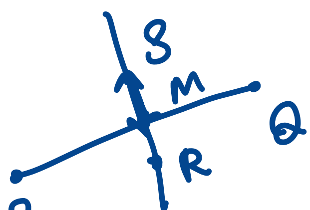
$$c = \frac{-27+50}{3} = \frac{23}{3}$$

$$y = -\frac{5}{3}x + \frac{23}{3}$$

$$= -\frac{35}{3} + \frac{23}{3} = -4$$

$$R\ (7, -4)$$

b. The point *S* lies on *l* such that its distance from *PQ* is 3 times the distance of *R* from *PQ*. Find the coordinates of the two possible positions of *S*. [3]



$$SM = 3MR$$

$$SM = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

$$= \sqrt{(y+9)^2 + (x-10)^2}$$

$$= \sqrt{\left(-\tfrac{5}{3}x + \frac{23}{3} + 9\right)^2 + (x-10)^2}$$

$$= \sqrt{\left(-\tfrac{5}{3}x + \frac{50}{3}\right)^2 + x^2 - 20x + 100}$$

$$= \sqrt{\frac{25}{9}x^2 - \frac{500}{9}x + \frac{2500}{9} + x^2 - 20x + 100}$$

$$= \sqrt{\frac{34}{9}x^2 - \frac{680}{9}x + \frac{3400}{9}}$$

$$MR = \sqrt{(-9+4)^2 + (10-7)^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34}$$

$$\sqrt{\frac{34}{9}x^2 - \frac{680}{9}x + \frac{3400}{9}} = 3\sqrt{34}$$

$$\frac{34}{9}x^2 - \frac{680}{9}x + \frac{3400}{9} = 9 \times 34$$
$$34x^2 - 680x + 3400 = 2754$$

$$34x^2 - 680x + 646 = 0$$

$x = 19$ or $x = 1$
$y = -24$ $\quad$ $y = 6$

7. The function f is defined by $f(x) = 2 - \sqrt{x + 5}$ for $-5 \leq x < 0$.

(i) Write down the range of f. [2]

$2 - \sqrt{0} = 2$ $\quad 2 - \sqrt{5} < y \leqslant 2$

$2 - \sqrt{5}$

(ii) Find $f^{-1}(x)$ and state its domain and range. [4]

$y = 2 - \sqrt{x+5}$

$\sqrt{x+5} = 2 - y$

$x + 5 = (2-y)^2$

$x = (2-y)^2 - 5$

$f^{-1}(x) = (2-x)^2 - 5, \quad 2 - \sqrt{5} < x \leqslant 2$

$-5 \leqslant y < 0$

The function g is defined by $g(x) = \frac{4}{x}$ for $-5 \leq x < -1$.

(iii) Solve $fg(x) = 0$. [3]

$g(x) = f^{-1}(0)$

$\frac{4}{x} = (2-0)^2 - 5$

$= 4 - 5$

$= -1$

$-4 = x$

8. Find constants a, b and c such that $\frac{\sqrt{p}q^{\frac{1}{3}}r^{-3}}{(\sqrt[5]{p}q^{-1})^{2}r^{-1}} = p^{a}q^{b}r^{c}$.

[3]

$$\frac{p^{1/2} q^{1/3} r^{-3}}{(p^{1/5} q^{-1})^{2} r^{-1}}$$

$$= \frac{p^{1/2} q^{1/3} r^{-2}}{p^{2/5} q^{-2}} = p^{1/10} q^{7/3} r^{-2}$$

$a = \frac{1}{10}$, $b = \frac{7}{3}$, $r = -2$

9. The polynomial $p(x) = 6x^{3} + ax^{2} + bx + 2$, where a and b are integers, has a factor of $x - 2$.

(a) Given that p(1) = –2p(0) , find the values of a and b.

[5]

$p(1) = 6 + a + b + 2$

$p(0) = 2$

$8 + a + b = -4$

$a + b = -12$ — 1

$p(2) = 48 + 4a + 2b + 2$

$-50 = 4a + 2b$

$-25 = 2a + b$

$-12 = a + b$

$-13 = a$

$a = -13$

$b = -12 - a$

$= -12 + 13$

$= 1$

(b) Using your values of *a* and *b*,

(i) find the remainder when p(*x*) is divided by 2*x* – 1.

$p(x) = 6x^3 - 13x^2 + x + 2$ [2]

$p(\frac{1}{2}) = \frac{3}{4} - \frac{13}{4} + \frac{1}{2} + 2$

$= 0$

(ii) factorise p(*x*) .

$p(x) = (x-2)(2x-1)(ax+b)$ [2]

$= (2x^2 - x - 4x + 2)(ax+b)$

$= (2x^2 - 5x + 2)(ax+b)$

$$
\begin{array}{r|l}
 & 3x + 1 \\
\hline
2x^2 - 5x + 2 & \cancel{6x^3} - 13x^2 + x + 2 \\
 & -\cancel{6x^3} \overset{+}{-} 15x^2 \overset{-}{+} 6x \\
\hline
 & 2x - 5x + 2
\end{array}
$$

$\therefore p(x) = (x-2)(2x-1)(3x+1)$

10.(a) Show that $\frac{\cos x}{1-\sin x} + \frac{1-\sin x}{\cos x} = 2\sec x$.

[5]

$$\frac{\cos^2 x + (1-\sin x)^2}{\cos x\,(1-\sin x)}$$

$$= \frac{\cancel{\cos^2 x} + 1 - 2\sin x + \cancel{\sin^2 x}}{\cos x\,(1-\sin x)}$$

$$= \frac{2-2\sin x}{\cos x\,(1-\sin x)} = \frac{2\,\cancel{(1-\sin x)}}{\cos x\,\cancel{(1-\sin x)}}$$

$$= \frac{2}{\cos x} = 2\sec x$$

$$= R.H.S$$

(b) Hence solve the equation $\frac{\cos\frac{\theta}{2}}{1-\sin\frac{\theta}{2}} + \frac{1-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = 8\cos^2\frac{\theta}{2}$ for $-360° < \theta < 360°$.

[5]

$-180° < \frac{\theta}{2} < 180°$

$$2\sec\frac{\theta}{2} = 8\cos^2\frac{\theta}{2}$$

$$\frac{2}{\cos\frac{\theta}{2}} = 8\cos^2\frac{\theta}{2}$$

$$\frac{1}{4} = \cos^3\frac{\theta}{2}$$

$$\cos\frac{\theta}{2} = \sqrt[3]{1/4}$$

$$\frac{\theta}{2} = \cos^{-1}\left(\sqrt[3]{1/4}\right)$$

$$\frac{\theta}{2} = 50.95°,\ -50.95°$$

$$\theta = 101.9,\ -101.9°$$